Journal of Statistical Physics, Vol. 61, Nos. 5/6, 1990

Hard-Hexagon Model: Calculation of Anisotropic Interfacial Tension from Asymptotic Degeneracy of Largest Eigenvalues of Row–Row Transfer Matrix

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Received May 23, 1990

To find the directional dependence of the interfacial tension of the hard-hexagon model, an inhomogeneous system is studied. This system is defined on a square lattice with $(1+v)M$ columns so that the lhs of the $(M+1)$th column is the hard-hexagon model and the rhs of the $(M+1)$th column works as the operator which shifts the particle configuration of a column downward. A triplet of the largest eigenvalues of the row–row transfer matrix are asymptotically degenerate as $M \to \infty$ under the conditions that $(1-v)M \equiv 0 \pmod 3$, with v being fixed to be constant. The interfacial tension of a tilted interface is calculated from the finite correction terms in this limit.

KEY WORDS: Hard-hexagon model; hard-square model; row–row transfer matrix; asymptotic degeneracy; interfacial tension.

Baxter and Pearce (BP) exactly calculated the interfacial tension of the hard-hexagon model for a special direction by two methods.[1] They considered a square lattice of M columns and N rows with toroidal boundary conditions. (A) When $M \equiv 0 \pmod 3$, they showed that a triplet of the largest eigenvalues of the row–row transfer matrix (RRTM) are asymptotically degenerate as $M \to \infty$, and calculated the interfacial tension from the finite correction terms. (B) When $M \equiv 1$ or $2 \pmod 3$, extra factors appear in the largest eigenvalues of the RRTM; BP pointed out that this fact reflects the existence of a mismatched vertical seam, and that the extra factors give the interfacial tension. In recent years, the crystal shape or interface physics has been attracted much attention. The calculation of the

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0022-4715/90/1200-1295$06 00/0

directional dependence of the interfacial tension is very important there. In a previous paper, we introduced the shift operator into the analysis (B) to obtain the anisotropic interfacial tension of the hard-hexagon model.[2] Furthermore, the equilibrium crystal shape was derived by the use of Wulff's construction. For the eight-vertex model, the calculation of the interfacial tension for a special direction was done by a method similar to (A).[3,4] It is significant to consider how the method (A) can be extended to the analysis of the anisotropic interfacial tension, since this extension is expected to be easily applicable to the eight-vertex model.

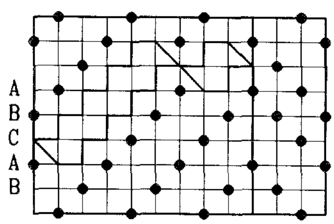
In this paper, we explain an alternative way of calculating the anisotropic interfacial tension of the hard-hexagon model, which is an extension of the method (A). In the first place, an inhomogeneous system is introduced. Next, the eigenvalues of the RRTM of this system are calculated by the commuting transfer matrix method, and it is shown that a triplet of the largest eigenvalues are asymptotically degenerate as the width of the system becomes large. The anisotropic interfacial tension of the hard-hexagon model is calculated from the finite correction terms in this limit.

The hard-hexagon model can be regarded as a special case of the hard-square model with diagonal interactions.[1,3] In the hard-square model an occupation number σ_i is located at each site i on the square lattice; $\sigma_i = 0$ if the site i is empty, and $\sigma_i = 1$ if the site i is occupied by a particle. Owing to the hard-core condition, a constraint $\sigma_i \sigma_j = 0$ is imposed on every nearest neighbor pair i, j. If the occupation numbers around a face are a, b, c, and d counterclockwise starting from the southwest corner, we assign a Boltzmann weight $W(a, b, c, d)$ on it. Baxter showed that this model is solvable when W's satisfy the star–triangle relation, which is a sufficient condition for commutability between RRTMs.[3,5] In this paper analysis is restricted to the triangular ordered phase corresponding to the regime II in BP. The parametrization of W's given by Eqs. (2.12), (3.10), and (3.14) in BP are used. We regard x as a real variable with $0 < x < 1$ and w as a complex one. The point $w = x^{-1}$ corresponds to the hard-hexagon model, and then x is related to the one-particle activity. The square lattice is interpreted as a deformed one of the triangular lattice where the hard-hexagon model is originally defined.

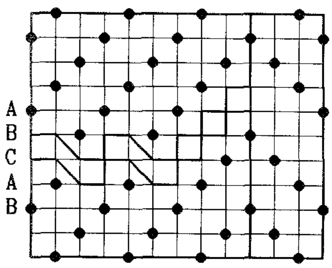
In BP the parameters x, w are common to all the faces. This condition is relaxed: w can vary from column to column.[3] The value of w between the ith column and the $(i+1)$th column is denoted by w_i. Consider a lattice of $(1+v)M$ $(0 < v < \infty)$ columns and N rows with toroidal boundary conditions, and set $w_1 = w_2 = \cdots = w_M = x^{-1}$, $w_{M+1} = w_{M+2} \cdots = w_{(1+v)M} = x$. The region $w = x$ works as the operator which shifts the configuration of a column downward. When M and N become large under the conditions that $(1-v)M \equiv 0 \pmod 3$ with v being fixed to be constant and $N \equiv 1$ or $2 \pmod 3$, there is a mismatched horizontal seam in the hard-hexagon region where $w = x^{-1}$ (Fig. 1). This seam is tilted due to the region $w = x$. The tilt angle is determined by v.

Restricting ourselves to near the interface (or the seam), we divide the lattice into three sublattices A, B, and C so that, on both sides of the interface, either the A lattice or the B lattice is preferentially occupied by particles. We call the phase where the A lattice (the B lattice) is preferentially occupied the A phase (the B phase). It is found that the positions of the A phase and the B phase are interchanged according as $N \equiv 1 \pmod 3$ or $N \equiv 2 \pmod 3$. In ref. 2, it was shown that the interfacial tension of these two types of interface are different. Taking account of this fact, we introduce a parameter $\theta_\perp$ by

$$-\frac{1}{v} = \frac{\sqrt{3}}{2} \tan \theta_\perp + \frac{1}{2},$$
$$\begin{cases} -\pi/2 < \theta_\perp < -\pi/6 & \text{for} \quad N \equiv 1 \pmod 3 \\ \pi/2 < \theta_\perp < 5\pi/6 & \text{for} \quad N \equiv 2 \pmod 3 \end{cases} \tag{1}$$



(a)



(b)

Fig. 1. Typical configurations of the inhomogeneous systems in the $x \to +0$ limit. (a) $N \equiv 2 \pmod 3$. (b) $N \equiv 1 \pmod 3$. The lhs of the tenth column is the hard-hexagon model, and the rhs of the tenth column works as the downward shift operator.

When the lattice is deformed into the triangular lattice, $\theta_\perp$ is the angle between the normal vector of the interface drawn from the A phase toward the B phase and the horizontal axis connecting the nearest neighbor sites. The interfacial tension is calculated as a function of $\theta_\perp$. Considering the case $w_1 = w_2 = \cdots = w_M = x^{-1}$, $w_{M+1} = w_{M+2} = \cdots = w_{(1+v)M} = 1$, we find the interfacial tension for $-\pi/6 < \theta_\perp < \pi/2$, $5\pi/6 < \theta_\perp < 3\pi/2$. Since the calculational methods are almost the same, only the calculation for $w_1 = w_2 = \cdots = w_M = x^{-1}$, $w_{M+1} = w_{M+2} = \cdots = w_{(1+v)M} = x$ is explained, and that for $w_1 = w_2 = \cdots = w_M = x^{-1}$, $w_{M+1} = w_{M+2} = \cdots = w_{(1+v)M} = 1$ is omitted in the following.

We consider a generalized problem where the value of w_i are given by a parameter w_0: $w_1 = w_2 = \cdots = w_M = w_0$, $w_{M+1} = w_{M+2} = \cdots = w_{(1+v)M} = x^2 w_0$. Hereafter, the new parameter w_0 is abbreviated to w. A one-parameter family of RRTMs is introduced. If $\sigma = \{\sigma_1, \sigma_2, ..., \sigma_{(1+v)M}\}$ and $\sigma' = \{\sigma'_1, \sigma'_2, ..., \sigma'_{(1+v)M}\}$ are the configurations of two successive rows, the RRTM is defined by

$$[\mathbf{V}_{IH}(w)]_{\sigma,\sigma'} = \prod_{i=1}^{M} W(\sigma_i, \sigma_{i+1}, \sigma'_{i+1}, \sigma'_i \mid w) \prod_{j=M+1}^{(1+v)M} W(\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma'_j \mid x^2 w) \quad (2)$$

where $\sigma_{(1+v)M+1} = \sigma_1$, $\sigma'_{(1+v)M+1} = \sigma'_1$. For convenience, we also define the dimensionless RRTM by

$$\mathbf{T}_{IH}(w) = \left(-\frac{\omega_1(w)}{\omega_4(w)\,\omega_5(w)}\right)^M \left(-\frac{\omega_1(x^2 w)}{\omega_4(x^2 w)\,\omega_5(x^2 w)}\right)^{Mv} \mathbf{V}_{IH}(w) \quad (2')$$

Unless otherwise mentioned, we regard x and v as constants. The transfer matrix argument in BP is repeated for $\mathbf{T}_{IH}(w)$. The family of $\mathbf{T}_{IH}(w)$ $[\mathbf{V}_{IH}(w)]$ commute with each other, being simultaneously diagonalized. The eigenvalues of $\mathbf{T}_{IH}(w)$ $[\mathbf{V}_{IH}(w)]$ are represented by $T_{IH}(w)$ $[V_{IH}(w)]$. It follows from the same derivation of (3.5) and (3.6) in BP that each eigenvalue $T_{IH}(w)$ satisfies the equations

$$T_{IH}(w)\, T_{IH}(xw) = 1 + T_{IH}(x^3 w) \quad (3)$$

$$T_{IH}(x^5 w) = T_{IH}(w) \quad (4)$$

Assuming some analytic properties of the eigenvalues $T_{IH}(w)$, and using (3) and (4), we can determine their asymptotic forms as $M \to \infty$. Hereafter, we confine ourselves to the eigenvalues which are the largest in the regime $1 \leqslant |w| \leqslant x^{-1}$.

First, the leading term of $T_{IH}(w)$ as $M \to \infty$ is considered. We always keep only the dominant term in the rhs of (3). In the $x \to +0$ limit, there is complete triangular order in the regimes $1 \leqslant |w| \leqslant x^{-1}$, $x^{5/2} \leqslant |w| \leqslant x^{3/2}$. This fact suggests that, in the $M \to \infty$ and the $x \to 0$ limit, the eigenvalues we consider behave as

$$V_{IH}(w) \sim w^{M/3} \left(\frac{1}{xw}\right)^{Mv/3}, \qquad 1 \leqslant |w| \leqslant x^{-1} \tag{5a}$$

It is also found that these eigenvalues are the largest in the regime $x^{5/2} \leqslant |w| \leqslant x^{3/2}$, and that, in the $M \to \infty$ and the $x \to \infty$ limit, they behave as

$$V_{IH}(w) \sim \left(\frac{x}{w}\right)^{M/3} \left(-\frac{1}{w^2}\right)^{Mv/3}, \qquad x^{5/2} \leqslant |w| \leqslant x^{3/2} \tag{5b}$$

In (5a) and (5b), the factors $w^{M/3}$, $(-1/w^2)^{Mv/3}$ correspond to the complete triangular order where the faces ω_3, ω_4 are dominant, and the factors $(1/xw)^{Mv/3}$, $(x/w)^{M/3}$ are related to the complete triangular order dominated by the faces ω_2, ω_5. We expect from (3), (5a), and (5b) that there exists a positive real number δ such that, for $0 < x < \delta$ and as $M \to \infty$,

$$\begin{aligned} &|T_{IH}(w)| = O(x^{-\varepsilon M}) \gg 1, \qquad x^{5/2} \leqslant |w| \leqslant x^{3/2}, \qquad 1 \leqslant |w| \leqslant x^{-1} \\ &|T_{IH}(w)| = O(x^{\varepsilon M}) \ll 1, \qquad x^{7/2} \leqslant |w| \leqslant x^{3}, \qquad x \leqslant |w| \leqslant x^{1/2} \end{aligned} \tag{6}$$

with $\varepsilon > 0$.

It is assumed that, except for exponential divergence as $M \to \infty$, $V_{IH}(w)$ has no infinity for $0 < |w| < \infty$. We also assume that the leading term of $V_{IH}(w)$ as $M \to \infty$ is analytic in the annuli $a < |w| < b$ containing the points $w = 1$, x^{-1} and $a' < |w| < b'$ containing the points $w = x^{5/2}$, $x^{3/2}$. In the limit $M \to \infty$, and for $0 < x < \delta$, it follows from (3) and (6) that the zeros of $V_{IH}(w)$ exist in the six annuli $x^{-1/2} < |w| < x^{-1}$, $x^{5/2} < |w| < x^2$, $1 < |w| < x^{-1/2}$, $x^2 < |w| < x^{3/2}$, $x^{-3/2} < |w| < x^{-2}$, $x < |w| < x^{1/2}$, and that the zeros $w = ax^{-1}$, ax^2 $(x^{1/2} < |a| < 1)$, and $w = bx^{-1/2}$, $bx^{3/2}$ $(x^{1/2} < |b| < 1)$ appear in pairs. It is found that, for $0 < x < \delta$ and M large, $T_{IH}(w)$ can be written in the form

$$\begin{aligned} T_{IH}(w) &= L(w)\, w^m \frac{\prod_{i=1}^{p}(1 - xw/a_i) \prod_{j=1}^{r}(1 - x^{1/2}w/b_j)}{(1 - xw)^M (1 - w^{-1})^{Mv}}, \\ &\qquad 1 \leqslant |w| \leqslant x^{-3/2} \\ T_{IH}(w) &= \bar{L}(w)\, w^{\bar{m}} \frac{\prod_{i=1}^{p}(1 - x^2 a_i/w) \prod_{j=1}^{r}(1 - x^{3/2} b_j/w)}{(1 - x^2/w)^{(1+v)M}}, \\ &\qquad x^{5/2} \leqslant |w| \leqslant x \end{aligned} \tag{7}$$

where $L(w)$ is analytic and nonzero for $1 < |w| < x^{-3/2}$ and $\bar{L}(w)$ is analytic and nonzero for $x^{5/2} < |w| < x$.

For the moment, we regard the a_i and b_j as known. Consider Eq. (3) in the annuli $x^{3/2} < |w| < x$, $x^{-1} < |w| < x^{-3/2}$, where the second terms in the rhs of (3) are dominant. Taking logarithms of both sides of (3), using (7), Laurent expanding, and equating coefficients, we can determine the explicit forms of $L(w)$ and $\bar{L}(w)$. It follows that p and r must satisfy the condition that $p + 2r \equiv (1-v)M \pmod 3$. We find that for $(p, r) = \mathbf{0} = (0, 0)$,

$$\begin{aligned} T_{IH;\mathbf{0},\tau}(w) &= \tau\psi(w)^M \psi(1/xw)^{Mv}, \qquad && 1 \leqslant |w| \leqslant x^{-3/2} \\ &= (1/\tau)\, \psi(x/w)^M \psi(w)^{Mv}, && x^{5/2} \leqslant |w| \leqslant x \end{aligned} \tag{8a}$$

with $\tau^3 = 1$, and that for $(p, r) \neq \mathbf{0}$,

$$\begin{aligned} T_{IH;p,r}(w) &= \psi(w)^M \psi(1/xw)^{Mv} \prod_{i=1}^{p} \psi(a_i/w) \prod_{j=1}^{r} \bar{\psi}(b_j/w), \\ &\qquad 1 \leqslant |w| \leqslant x^{-3/2} \\ &= \psi(x/w)^M \psi(w)^{Mv} \prod_{i=1}^{p} \psi(w/xa_i) \prod_{j=1}^{r} \bar{\psi}(w/xb_j), \\ &\qquad x^{5/2} \leqslant |w| \leqslant x \end{aligned} \tag{8b}$$

where the a_i and b_j are defined on the three sheets of the Riemann surface. The definitions of $\psi(w)$ and $\bar{\psi}(w)$ are given by (5.8) in BP. The facts that $|\psi(w)| > 1$ for $1 < |w| < x^{-3/2}$, $|\psi(w)| < 1$ for $x^{3/2} < |w| < 1$, and $\psi(x^3 w) = \psi(w)$ show that the conditions (6) are satisfied for $0 < x < 1$. Therefore, the argument from (6)–(8) makes sense for $0 < x < 1$. The leading term of $T_{IH}(w)$ for $x < |w| < 1$, $x^{7/2} < |w| < x^{5/2}$ can be determined by the use of (8a), (8b), and (3).

The a_i and b_j are solutions of the equations

$$\begin{aligned} \psi(a_i)^M \psi(1/a_i x)^{Mv} &= -\prod_{k=1}^{p} \psi(a_i/a_k) \prod_{l=1}^{r} \bar{\psi}(a_i/b_l), \qquad && i = 1, 2,\dots, p \\ \bar{\psi}(b_j)^M \bar{\psi}(1/b_j x)^{Mv} &= -\prod_{k=1}^{p} \bar{\psi}(b_j/a_k) \prod_{l=1}^{r} \psi(b_j/b_l), && j = 1, 2,\dots, r \end{aligned} \tag{9}$$

Equations (9) show that as $M \to \infty$, the a_i and b_j approach the contours $|\psi(a)\, \psi(1/ax)^v| = 1$ and $|\bar{\psi}(b)\, \bar{\psi}(1/bx)^v| = 1$, respectively. This fact is consistent with the requirements that $x^{1/2} < |a_i| < 1$, $x^{1/2} < |b_j| < 1$. We find that, when $(1-v)M \equiv 0 \pmod 3$, the triplet of eigenvalues $T_{IH;\mathbf{0},\tau}(w)$ are the largest in the regimes $1 \leqslant |w| \leqslant x^{-1}$, $x^{5/2} \leqslant |w| \leqslant x^{3/2}$.

Next, for the triplet of the largest eigenvalues $T_{IH;\mathbf{0},\tau}(w)$, an integral equation determining the finite correction terms as $M \to \infty$ is derived. In this calculation, we keep both terms in the rhs of (3). We define $K_\tau(w)$, $\bar{K}_\tau(w)$ by

$$\begin{aligned} T_{IH;\mathbf{0},\tau}(w) &= \tau\psi(w)^M \psi(1/xw)^{Mv} K_\tau(w), \qquad & 1 \leqslant |w| \leqslant x^{-3/2} \\ &= (1/\tau)\,\psi(x/w)^M \psi(w)^{Mv} \bar{K}_\tau(w), \qquad & x^{5/2} \leqslant |w| \leqslant x \end{aligned} \tag{10}$$

It follows from (3) that

$$\begin{aligned} \frac{K_\tau(w)\,K_\tau(xw)}{\bar{K}_\tau(x^3 w)} &= 1 + \frac{1}{T_{IH;\mathbf{0},\tau}(x^3 w)}, \qquad & x^{-1} < |w| < x^{-3/2} \\ \frac{\bar{K}_\tau(w)\,\bar{K}_\tau(xw)}{K_\tau(x^{-2} w)} &= 1 + \frac{1}{T_{IH;\mathbf{0},\tau}(x^{-2} w)}, \qquad & x^{3/2} < |w| < x \end{aligned} \tag{11}$$

For sufficiently large M, the second terms in the rhs of (11) are exponentially smaller than 1. Taking logarithms of both sides of (11), Laurent expanding, and equating coefficients, we get the integral equation

$$\begin{aligned} \ln K_\tau(w) = &-\frac{1}{2\pi i}\oint_{C_1} \frac{dw'}{w'} \ln\left[1 + \frac{1}{T_{IH;\mathbf{0},\tau}(x^3 w')}\right] J\left(\frac{xw}{w'}\right) \\ &+\frac{1}{2\pi i}\oint_{C_2} \frac{dw'}{w'} \ln\left[1 + \frac{1}{T_{IH;\mathbf{0},\tau}(x^{-2} w')}\right] J\left(\frac{x^2 w}{w'}\right) \end{aligned} \tag{12}$$

where C_1 is a circle in $x^{-1} < |w'| < x^{-3/2}$, C_2 is a circle in $x^{3/2} < |w'| < x$, and $J(w)$ is the function defined by (5.15) in BP. For $1 \leqslant |w| \leqslant x^{-3/2}$, Eqs. (10) and (12) determine the asymptotic form of $T_{IH;\mathbf{0},\tau}(w)$ as $M \to \infty$.

For large M, using (8a), we estimate the logarithms in the integrands of (12) by

$$\begin{aligned} \ln[1 + 1/T_{IH;\mathbf{0},\tau}(x^3 w')] &\sim \tau[\psi(w'/x)\,\psi(1/w')^v]^M \\ \ln[1 + 1/T_{IH;\mathbf{0},\tau}(x^{-2} w')] &\sim (1/\tau)[\psi(x^2/w')\,\psi(w'/x)^v]^M \end{aligned} \tag{13}$$

and integrate (12) by steepest descent. It follows that

$$\begin{aligned} K_\tau(w) = 1 &+ \alpha(w)\,\tau\psi(w_s/x)^M \psi(1/w_s)^{Mv} \\ &+ \bar{\alpha}(w)(1/\tau)\,\psi(x^2/\bar{w}_s)^M \psi(\bar{w}_s/x)^{Mv} + \cdots \end{aligned} \tag{14}$$

where w_s and $\bar{w}_s$ are the saddle points of $|\psi(w'/x)\,\psi(1/w')^v|$ and $|\psi(x^2/w')\,\psi(w'/x)^v|$, respectively. When w_s and $\bar{w}_s$ are regarded as functions of v, they satisfy the conditions that $w_s = -x^{-1}$, $\bar{w}_s = -x^{3/2}$ for $v = 1$. The functions $\alpha(w)$, $\bar{\alpha}(w)$ are represented by $J(w)$ and the derivatives of $\psi(w)$, and their explicit forms are not important here. For $(1-v)M \equiv 0 \pmod 3$ and $1 \leqslant |w| \leqslant x^{-1}$, Eqs. (10) and (14) show that the triplet of the largest eigenvalues $T_{IH;0,\tau}(w)$ are asymptotically degenerate as $M \to \infty$.

Now, setting $w = x^{-1}$ in (10) and (14), we calculate the anisotropic interfacial tension of the hard-hexagon model. When $(1-v)M \equiv 0 \pmod 3$ and M, N become large, the partition function can be represented by the use of (10) and (14) as

$$\begin{aligned} Z \sim \Bigg[& \left(1 + \tau_0^N + \frac{1}{\tau_0^N}\right) + \alpha(x^{-1}) N \left(1 + \tau_0^{N+1} + \frac{1}{\tau_0^{N+1}}\right) \\ & \times \psi\left(\frac{w_s}{x}\right)^M \psi\left(\frac{1}{w_s}\right)^{Mv} \\ & + \bar{\alpha}(x^{-1}) N \left(1 + \tau_0^{N-1} + \frac{1}{\tau_0^{N-1}}\right) \\ & \times \psi\left(\frac{x^2}{\bar{w}_s}\right)^M \psi\left(\frac{\bar{w}_s}{x}\right)^{Mv} \Bigg] \kappa^{MN} \end{aligned} \tag{15}$$

where $\tau_0 = (-1 + \sqrt{3}\, i)/2$ and

$$\kappa = \left(-\frac{\omega_4(w)\, \omega_5(w)}{\omega_1(w)}\right) \psi(w), \qquad w = x^{-1} \tag{16}$$

The second and the third terms in the bracket of (15), which come from the finite correction terms in (14), give the excess free energy for $N \equiv 2$ and 1 (mod 3), respectively. From (14) and (15), after some calculations, we find that

$$\begin{gathered} -\beta\sigma = \frac{2}{\sqrt{3}} \left[\cos\theta_\perp \ln |\psi(a_s x)| + \cos\left(\theta_\perp - \frac{\pi}{3}\right) \ln |\psi(a_s)| \right] \\ \text{for} \quad -\frac{\pi}{2} < \theta_\perp < -\frac{\pi}{6}, \quad \frac{\pi}{2} < \theta_\perp < \frac{5\pi}{6} \end{gathered} \tag{17}$$

where β is the inverse temperature and σ is the interfacial tension defined on the triangular lattice. The saddle point a_s is determined by

$$\frac{\sqrt{3} - \tan\theta_\perp}{\sqrt{3} + \tan\theta_\perp} = -a \frac{f(-ax, x^3)\, f(a^{-1}x^{1/2}, x^3)\, f(-a^{-1}x^{1/2}, x^3)}{f(-a^{-1}x, x^3)\, f(ax^{1/2}, x^3)\, f(-ax^{1/2}, x^3)}, \qquad a = a_s \tag{18a}$$

and the conditions

$$a_s = -x^{1/2}, \qquad \theta_\perp = -\frac{\pi}{3}$$
$$a_s = -x, \qquad \theta_\perp = \frac{2\pi}{3} \tag{18b}$$

Combining the result of the case $w_1 = w_2 = \cdots = w_M = x^{-1}$, $w_{M+1} = w_{M+2} = \cdots = w_{(1+v)M} = 1$ with (17), (18a), and (18b), we obtain an expression of the interfacial tension for all directions which is the same result as that given in Section 3.2 of ref. 2.

To summarize, we have shown that the anisotropic interfacial tension of the hard-hexagon model can be calculated from the finite correction terms of the triplet of the largest eigenvalues of the RRTM of the inhomogeneous system, which are asymptotically degenerate as $M \to \infty$. As is mentioned in the beginning of this paper, this calculation is applicable to a wide class of solvable models, including the eight-vertex model. Recently, Holzer[6] and Akutsu and Akutsu[7] have shown that, for the planar Ising model without bond crossings, the equilibrium crystal shape is given by a set of imaginary zeros of the partition function, and that this property originates from the free random-walk character of the interface. Akutsu and Akutsu also examined the expression of the facet shape of the BCSOS model (or the six-vertex model),[8] and suggested the free random-walk character of the step of this model. In connection with these problems, it is desirable to carry out the analysis of the anisotropic interfacial tension of the eight-vertex model, which contains the square lattice Ising model and the six-vertex model as special limits.[3] I hope to explore this problem in a future publication.

ACKNOWLEDGMENTS

This work originated from a discussion with Dr. S. Sasaki, which is gratefully acknowledged. I also wish to thank Prof. P. A. Pearce for sending a detailed explanation of ref. 1.

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Communicated by J. L. Lebowitz